RADICAL PROPERTIES OF CERTAIN GROUP ALGEBRAS.

R. P. KNOTT.

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Abstract.

In this thesis we are concerned with the following problem: if F is a field and G a group, what conditions must be imposed on F and G so that the group algebra of G over F is $\mathcal{S}$ semi-simple? Here, $\mathcal{S}$ is one of the ring properties, nil, nilpotent, right-quasi-regular or B-regular.

In the first few chapters we survey the known conditions where $\mathcal{S}$ is nil, nilpotent or right-quasi--regular. In a later chapter we shall show that if F is any field, the group algebra of G over F is right-quasi--regular semi-simple if G belongs to a certain class of torsion-free generalised soluble groups. We then study a generalisation of the group algebra, namely the twisted group algebra, and determine conditions on F and G when $\mathcal{S}$ is nil, nilpotent or right-quasi-regular. Finally we assume that $\mathcal{S}$ is B-regularity. We then determine conditions on F and G for the group algebra to be B--regular semi-simple when G belongs to a certain class of generalised nilpotent groups.

Chapter 1. Introduction and prerequisites.

1. One of the best known theorems from the theory of representations of finite groups is the following consequence of Maschke's theorem: if G is any finite group and F any field of characteristic p where, if $p \neq 0$, G has no elements of order p, then FG, the group algebra of G over F has zero radical.

We recall that under these conditions on G and F, FG is an algebra with both the ascending and descending chain conditions on right ideals. We shall study the group algebra of an arbitrary group over any field F and attempt to prove the generalisation of this theorem.

If R is any ring with both the descending and ascending chain conditions, various radical properties are equivalent. Thus R has a unique radical. Therefore when we generalise the above theorem we must be more specific about which radical we are referring to.

We shall use FG to denote the group algebra of G over F.

In chapter 2 we shall study the nil radical and will determine sufficient conditions for FG to have zero nil radical. The results of this chapter are taken from [18].

Chapter 3 contains most of the known results for the Jacobson radical. We are unable to determine a complete solution for FG to have zero Jacobson radical but will show that FG has zero Jacobson radical for any $\mathfrak{X}$-group and F any field of characteristic p where p is determined by G and $\mathfrak{X}$. We shall show that $\mathfrak{X}$ can take the values $\mathfrak{F}$, $\mathfrak{A}$, $R\mathfrak{F}$, $\hat{E}\mathfrak{A}$.

In chapter 4 we shall study a related problem. If G is any torsion-free group and F any field is the Jacobson radical of FG zero? Again, we are unable to obtain a complete solution but will show that FG has zero Jacobson radical for any field F, when G is a torsion-free group belonging to either the class $\hat{E}\mathfrak{A}^*$ or the class $\acute{E}L\mathfrak{N}$. The latter result is contained in our forthcoming paper [12].

In chapter 5 we make another generalisation. There we are concerned with the twisted group algebra F^tG. We shall show that most of the results of chapters 2 and 3 are still true when FG is replaced by F^tG. The twisted group algebra is the object of study by Passman in a series of recent papers from which many of the results of this chapter are taken.

In the final chapter we take our radical to be the Brown-McCoy radical. We shall show that FG has zero Brown-McCoy radical, for a field F of characteristic p where p is determined by G, if G is any ZA-group. We shall also show that FG has zero Brown-McCoy radical for any field F if G is a torsion-free ZA-group. We shall also give a few results for metabelian groups. However, we have been unable to determine that FG has zero Brown-McCoy radical for G any metabelian group and F any field. of suitable characteristic. However we feel that the partial results we have obtained are sufficient evidence for us to conjecture that such a solution is possible.

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The rest of this chapter contains a few results and definitions from group and ring theory that we consider prerequisites for the remainder of this thesis.

2. Groups.

We shall need a working knowledge of group theory, supplemented by the following definitions.

Let G be a group. A <u>series</u> in G is a set of subgroups, $\underline{S} = \{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ where Σ is a linearly ordered set such that for all $\tau, \sigma \in \Sigma$,

(i) $V_\sigma \lhd \Lambda_\sigma$,

(ii) $\Lambda_\sigma \leq V_\tau$, $\sigma < \tau$,

(iii) $G - \{1\} = \bigcup_{\sigma \in \Sigma} (\Lambda_\sigma - V_\sigma)$.

$\Lambda_\sigma - V_\sigma$ is a <u>layer</u> of G and $\Lambda_\sigma / V_\sigma$ is a <u>factor</u> of G. It is a consequence of these definitions that each element $\neq 1$ of G belongs to a unique layer of G and that $V_\sigma = \bigcup_{\lambda < \sigma} \Lambda_\lambda$, $\Lambda_\sigma = \bigcap_{\sigma < \lambda} V_\lambda$.

$\underline{S}$ is a <u>normal series</u> of G if $\Lambda_\sigma, V_\sigma \lhd G$ for all $\sigma \in \Sigma$.

If Σ is a well ordered set we can take $\Sigma = \rho$, an ordinal number. In this case

$\Lambda_\sigma = \bigcap_{\lambda > \sigma} V_\lambda = V_{\sigma+1}$ if $\sigma+1 < \rho$ and $\Lambda_\sigma = G$ if $\sigma+1 = \rho$. Then ~~Let~~ $V_0 = \{1\}$ and $V_\rho = G$.

Then the Λ_σ's are superfluous and $\underline{S}$ is an <u>ascending series</u> of G. If Σ is a finite set $\underline{S}$ is a finite series; i.e. for some n there exists a series $\{1\} = V_0 \leqslant V_1 \leqslant \dots \leqslant V_n = G$, where V_i is normal in V_{i+1} for all $i = 1, 2, \dots, n-1$.

Let x,y be elements of G. We define $\underline{[x,y]}$, the commutator of x and y by $[x,y] = x^{-1}y^{-1}xy$. Let X, Y be subsets of G. We define $\underline{[X,Y]}$ as the subgroup of G generated by the $[x,y]$ for $x \in X$ and $y \in Y$. A series is central if $[\Lambda_\sigma, G] \leqslant V_\sigma$ for all $\sigma \in \Sigma$.

Let X be any set. We define $\underline{\langle X \rangle}$ as the subgroup of G generated by the set X. If X is a finite set, say $X = \{x_1, x_2, \dots, x_n\}$, we shall write $\langle x_1, x_2, \dots, x_n \rangle$ for $\langle X \rangle$.

We shall use the closure operations and class notation introduced by P. Hall in [9]. We define $\mathfrak{X}$ to be a class of groups if (i) $G \cong G_1 \in \mathfrak{X}$ implies that $G \in \mathfrak{X}$ and (ii) $\mathfrak{X}$ contains a group of order 1. Examples of the types of classes that will concern us are,

$\mathfrak{F}$ = finite groups.

$\mathfrak{G}$ = finitely generated groups.

$\mathfrak{F}_\pi$ = finite π -groups for π a set of primes.

$\mathfrak{N}$ = nilpotent groups.

$\mathfrak{A}$ = abelian groups.

Let A be a function on the class of all classes of groups taking values in that class: A is a <u>closure operation</u> if $A(1) = (1)$ where (1) is the class of unit groups and if $\mathfrak{X} \leq A\mathfrak{X} = A^2\mathfrak{X} \leq A\mathfrak{Y}$ whenever $\mathfrak{X} \leq \mathfrak{Y}$.

A class $\mathfrak{X}$ is <u>A-closed</u> if $A\mathfrak{X} = \mathfrak{X}$.

Familiar examples of closure operations are S, Q, E, L, R, $\hat{E}$, $\acute{E}$ defined as follows:

Let $\mathfrak{X}$ be a class of groups. G belongs to $\underline{S\mathfrak{X}}$ if and only if G is a subgroup of an $\mathfrak{X}$-group. $G \in \underline{Q\mathfrak{X}}$ if and only if G is the homomorphic image of an $\mathfrak{X}$-group. $G \in \underline{E\mathfrak{X}}$ if and only if G ~~is the extension of a $\mathfrak{X}$-group by a $\mathfrak{X}$-group.~~ has a series with $\mathfrak{X}$-factors. G belongs to $\underline{L\mathfrak{X}}$ if and only if every finite subset of G is contained in some $\mathfrak{X}$-group of G. $G \in \underline{R\mathfrak{X}}$ if and only if G is residually an $\mathfrak{X}$-group. That is if and only if for each x ($\neq 1$) in G, there exists a normal subgroup N of G such that $x \notin N$ and $G/N \in \mathfrak{X}$. G belongs to $\underline{\hat{E}\mathfrak{X}}$ if and only if G has a series in which each factor is an $\mathfrak{X}$-group. Finally, G belongs to $\underline{\acute{E}\mathfrak{X}}$ if and only if G has an ascending series in which each factor is a $\mathfrak{X}$-group.

3. Rings and radicals.

The main source for this section is Divinsky [7]. Let $\mathcal{S}$ be a certain property that a ring may possess. An ideal of a ring will be called a $\mathcal{S}$ -ideal if it, as a ring, possesses the property $\mathcal{S}$. A ring which does not possess any non zero $\mathcal{S}$ -ideals is defined to be $\mathcal{S}$ -semi-simple.

We shall call $\mathcal{S}$ a radical property if the following three conditions hold:

(a) every homomorphic image of a $\mathcal{S}$ -ring is a $\mathcal{S}$ -ring.

(b) every ring R contains a maximal $\mathcal{S}$ -ideal, denoted by S, that contains every other $\mathcal{S}$ -ideal of R.

(c) R/S is $\mathcal{S}$ -semi-simple.

The maximal $\mathcal{S}$ -ideal of R is called the $\mathcal{S}$ -radical of the ring R.

The radical properties which will concern us are those defined as nil, right-quasi-regular and B-regular. The right-quasi-regular and B-regular radicals are defined as the Jacobson and Brown-McCoy radical respectively.

Let R be any ring and x an element of R. x is defined to be nilpotent if there exists a positive integer n such that $x^n = 0$.

A ring is defined to be nil if every element of the ring is nilpotent. A ring is defined to be nilpotent if there exists a positive integer m such that $R^m = \{0\}$, where R denotes the ring. With these definitions, nil is a radical property. We shall use N(R) to denote the nil radical of the ring R.

However, the join of all the nilpotent ideals in any ring R is a nil, but not necessarily nilpotent, ideal of R. Thus, nilpotency is not a radical property. However, it is customary to refer, in some sections of the literature, to the nilpotent radical. We shall avoid this usage and use $W(R)$ to denote the union of all the nilpotent ideals of R.

Let x,y be any two elements of some ring R. We define a binary operation '$\circ$' on R by $x \circ y = x + y + xy$. We say x is <u>right-quasi-regular</u> if there exists a y such that $x \circ y = 0$. We shall often shorten right-quasi-regular to r.q.r.. In the above y is defined to be the <u>right-quasi--inverse</u> of x in R. In a similar way we can define left-quasi-regular and left-quasi-inverse.

If x in nilpotent, say $x^n = 0$, then is is r.q.r. with right-quasi-inverse $y = -x + x^2 - x^3 + + \dots \pm x^{n-1}$. If R is a ring with a 1, x is r.q.r. if and only if $1 + x$ is a unit of R. Clearly, $x \circ y = 0$ if and only if $(1 + x)(1 + y) = 1$. Let $J(x) = \{xr + r : r \in R\}$. Then $J(x)$ is a right ideal of R and x is r.q.r. if and only if $J(x) = R$.

We say that a ring is right-quasi-regular if every element of it is r.q.r.. Let $J(R)$ denote the join of all the right-quasi-regular right ideals of R. Then $J(R)$ is a two sided ideal and $R/J(R)$ contains no non zero r.q.r. right ideals. Also, $J(R)$ is equal to the union of all the right-quasi-regular left ideals of R and $R/J(R)$ contains no non zero right-quasi-regular left ideals. Thus, right-quasi-regularity is a radical property and $J(R)$, the right-quasi-regular radical, is called the Jacobson radical. R is defined to be <u>Jacobson semi-simple</u> if $J(R) = \{0\}$.

We shall give some alternative definitions of $J(R)$. Clearly we could replace right-quasi-regularity by left-quasi-regularity. Thus $J(R)$ is the union of all the left-quasi-regular right (left) ideals of R. Note that if $x \circ y = 0$ then y is left-quasi-regular. Thus, $x \in J(R)$ if and only if its right-quasi-inverse, y, also belongs to $J(R)$.

We define an ideal V of R to be <u>regular</u> if and only if there exists an $e \in R$ such that for all $r \in R$, $er - r \in V$. Then $J(R)$ is equal to the intersection of all the maximal regular right ideals of R. If R has a 1, every ideal is regular.

$J(R)$ is also equal to the intersection of all the kernels of all the irreducible representations of R.

We have noted that $J(x)$ is a right ideal of R and $J(x) = R$ if and only if x is right-quasi-regular. Let $B(x)$ denote the two sided ideal of R generated by $J(x)$. That is, $B(x) = \{xr + r + \sum x_i x y_i + x_i y_i$ where $x_i, y_i \in R$ and sum is finite$\}$. We define an element x of R to be <u>B-regular</u> if $B(x) = R$. An ideal I of R is defined to be <u>B-regular</u> if and only if every element of I is B-regular.

B-regularity is a radical property and the maximal B-regular ideal of R is called the Brown-McCoy radical of R, denoted by B(R). R is defined to be <u>Brown-McCoy semi-simple</u> if $B(R) = \{0\}$.

If R has a 1, B(R) is also equal to the intersection of all the maximal two sided ideals of R. R/B(R) is always a subdirect sum of simple rings with a 1.

Clearly each right-quasi-regular ideal is B-regular. Thus we can clearly see that the three radicals satisfy the following inclusions: $N(R) \leqslant J(R) \leqslant B(R)$. All three are equal if R has the descending chain condition on right (left) ideals. Further it is clear that $J(R) = B(R)$ if R is commutative.

We recall that if R is a ~~group~~ ring with the descending chain condition on right ideals $W(RG) = N(RG)$.

4. The group algebra.

We begin with the definition of an algebra. Let A be a set with the operations of addition and multiplication defined in such a way as to make A into a ring. Suppose further that we can define an operation of ~~right~~ multiplication on A by the elements of some field F so that A is a vector space over F. Assume also that if $c \in F$ and $x, y \in A$ then these operations are connected by the relations $c(xy) = cx(y) = x(cy)$. Then A is an <u>algebra</u> over F.

The main example we shall be concerned with is the group algebra. Let G be a group and F any field. Let FG denote the set of all formal sums $\sum_{g \in G} \alpha_g g$, where $\alpha_g \in F$ and all but a finite number are zero. We define addition component-wise to make FG into a vector space over F, with the group elements as a basis. Let $a = \sum_{g \in G} \alpha_g g$

$a' = \sum_{h \in G} \beta_h h$. Then we define multiplication

in FG by $aa' = \sum_{g \in G} \sum_{h \in G} \alpha_g \beta_h gh$.

Then with these definitions FG is an algebra over F called the group algebra of G over F. Clearly this algebra has a unit with respect to multiplication.

We define a subalgebra and a right ideal to be a vector subspace closed under multiplication and right multiplication by the elements of FG respectively. Thus if I is a right ideal we must have $IF \leq I$. We could also consider FG as a ring and define a subring and right ideal in the usual way. Thus, it is clear that a ring ideal might not necessarily be an algebra ideal. Since we are concerned with radicals the following theorem is relevant:

Theorem ([7] theorem 39.) If A is an algebra over a field F and $\mathcal{S}$ is a radical property, then the $\mathcal{S}$-radical of A thought of as an algebra, exists, and is equal to the $\mathcal{S}$-radical of A, thought of merely as a ring.

Hence, we shall not usually distinguish between a ring or an algebra radical.

The definitions of left and two sided ideals are the obvious ones, bearing in mind our definition of a right ideal.

Let $x \in FG$ and assume that $x = \sum_{g \in G} \alpha_g g$.

We define the <u>Support of x</u>, written as <u>Supp(x)</u> by

$\mathrm{Supp}(x) = \{ g \in G : \alpha_g \neq 0 \}$. Thus, for all

$x \in FG$, Supp(x) is a finite set and we shall denote the cardinality of this set by $|\mathrm{Supp}(x)|$ called the <u>length</u> of x.

Let H be a subgroup of G and T a right transversal to H in G. We shall always assume that T contains 1 to represent the coset H. Let x be any non zero element of FG. Then x can be written uniquely in the form

$$x = \sum_{i=1}^{n} \alpha_i g_i, \text{ where } n \geqslant 1,$$

$0 \neq \alpha_i \in FH$ and the g_i are distinct elements of T for $i = 1, 2, \ldots, n$.

We define this as the <u>H-decomposition</u> of x. The integer n is called the <u>H-length</u> of x. Note that if $g \in G$, x and xg have the same H-length. If H is normal in G then gx and x have the same H-length, for all $g \in G$. ~~Thus, if FG has an element of fixed H-length n then FG has at least n elements of H-length n and such that their support has non zero intersection with H.~~

~~If x is as above then xg_i^{-1} for $i = 1, 2, \ldots, n$, are the elements refered to.~~ Clearly in the H-decomposition of xg_i^{-1} one of the g_i's is 1 and thus $\mathrm{Supp}(xg_i^{-1}) \cap H \neq \emptyset$.

Chapter 2. The nil radical and nilpotent ideals.

The main source for this chapter is Passman [18]. We begin, however, with a result due to Amitsur [2].

Lemma 2.1. Let Q be the rational field and G any group. Then QG is nil semi-simple.

Passman was able in [18] to prove the obvious generalisation of this result.

Lemma 2.2. Let F be any field of characteristic zero and G any group. Then FG is nil semi-simple.

For the case of F, a field of characteristic p, he proved the following lemma.

Lemma 2.3. If G is a group with no elements of order p and F any field of characteristic p then FG is nil semi-simple.

The above lemma gives sufficient conditions for the nil radical of a group algebra to be zero.

We shall give an example to show that these conditions are not, however, necessary. Let A be an abelian group with no elements of even order and let X be the group $\{1,t\}$ of order 2. Take G to be the split extension of A by X where t acts on A by sending each element to its inverse.

Lemma 2.4. If G is the group described above and F any field of characteristic 2. Then:

(i) if A is finite, $N(FG) = F.\sum_{g \in G} g$.

(ii) if A is infinite, $N(FG) = \{0\}$.

This example is due to Passman [18]. In a later chapter we shall see that if A is torsion-free, FG has zero Brown-McCoy radical.

In the same paper Passman was also able to give the following necessary and sufficient conditions for FG to have no non zero nilpotent ideals.

Theorem 2.5. Let G be a group and F any field of characteristic p. If p = 0, FG has no non zero nilpotent ideals. If $p \neq 0$, FG has no non zero nilpotent ideals if and only if G has no finite normal subgroups of order divisible by p.

Thus, we are able to determine necessary and sufficient conditions for W(FG) to be zero. We are able to determine sufficient conditions for N(FG) to be zero. In the next section we shall show that it is far harder to make such statements about the Jacobson radical.

Chapter 3. The Jacobson radical.

In this chapter we shall show under what conditions we can determine a solution to an open question due to Amitsur. The question is: if G is a group and F any field of characteristic zero, is FG Jacobson semi-simple?

The first solution to this question was given by C.E.Rickart in [23] using the methods of Banach algebras. His result was:

Theorem 3.1. Let G be any group and F either the real or complex field. Then FG is Jacobson semi-simple.

Amitsur again placed emphasis on the field in [1] where he first proved our next two theorems. Before stating them however, we must introduce some new notation.

Let F be a field, K an extension field of F and A an algebra over F. Then $\underline{A_K} = A \otimes_F K$.

Theorem 3.2. Let K be a separable extension of F (of finite or infinite degree). Then $J(A_K) = (J(A))_K$.

Theorem 3.3. Let K be a pure transcendental extension of F. Then $J(A_K) = N_K$, where N is equal to $J(A_K) \cap A$ and is a nil ideal of A.

Corollary 3.4. ([1]) If QG is Jacobson semi-simple for Q the rational field then FG is Jacobson semi-simple for any field F of characteristic 0.

Corollary 3.5. ([1]) Let F be a field of characteristic zero and assume that F is a transcendental extension of the rationals. Then FG is Jacobson semi-simple.

Proofs of both these corollaries follow from
Theorem 3.2
~~putting A = FG and using lemma 2.1.~~

Passman has shown in [18] a similar result for F any field of characteristic $p \neq 0$.

Corollary 3.6. Let F be a field of characteristic $p \neq 0$ where F is a separably generated non-algebraic extension of some subfield K. Then if G has no elements of order p, FG is Jacobson semi-simple.

The final result in this direction was recently proved in [21] by Passman. We define an algebra A over a field F to be <u>nilpotent-free</u> if A_K has no non zero nilpotent ideals for any extension field K of F.

Theorem 3.7. Let A be a nilpotent-free algebra over a field F and let K be an algebraic extension of F. Then $J(A) = \{0\}$ if and only if $J(A_K) = \{0\}$.

Using the results of chapter 2 we are able to see that FG is nilpotent-free for any group G if F is a field of characteristic zero and for all groups G such that G has no finite normal subgroups of order divisible by p if F is a field of characteristic $p \neq 0$.

One consequence of these theorems is that we can assume the field, over which we take our group algebra, is algebraically closed. For if we consider the group algebra of a group G over a field F, FG is Jacobson semi-simple if and only if KG is Jacobson semi-simple, where K is the algebraic closure of F. This technique is useful especially in the proof of the next lemma and also in the proof of theorem 5.10. In both cases the proof depends on the algebraic closure of the field concerned.

Theorem 3.8. (Theorem 5 of [19])
Let G be a group, H a normal subgroup such that G/H is abelian. Let F be an algebraically closed field of characteristic p where, if $p \neq 0$, G/H has no elements of order p. Then if I is any characteristic ideal of FG, $I = (I \cap FH)FG$.

The proofs of theorems 3.2, 3.3 and 3.7 all use certain properties of fields. This type of approach, however, has proved so far to be insuffIcent to give a complete solution to the problem. In the remainder of this chapter we shall show that it is possible to obtain a solution, provided we put a certain structure (usually a finite or solubility condition), on the group. If G belongs to a class $\mathfrak{X}$, and F is any field of characteristic $p \in \pi$ where π is a set of primes determined by G and $\mathfrak{X}$. Then FG is Jacobson semi-simple where $\mathfrak{X}$ is $\mathfrak{F}$, $\mathfrak{A}$, $\hat{E}\mathfrak{A}$ or $R\mathfrak{F}$.

If $G \in \mathfrak{F}$ then the following is an immediate consequence of Maschke's theorem (see for example [6]).

Theorem 3.9. Let G be a finite group and F any field of characteristic p where, if $p \neq 0$, G has no elements of order p. Then FG is Jacobson semi-simple.

If $G \in \mathfrak{A}$ then we can use theorem 3.7 and set $I = J(FG)$ and $H = \langle 1 \rangle$ in theorem 3.8 to show that FG is Jacobson semi-simple for any field F of characteristic p, provided that G has no elements of order p if $p \neq 0$.

Before discussing the class $\hat{E}\mathfrak{A}$ we shall pause to prove a lemma of our own that is a generalisation of a result of Amitsur contained in [2]. This is not idle generalisation as we use this lemma in its full generality in chapter 6.

Lemma 3.10. Let G be any group, H a normal subgroup and F any field. Let S be any subalgebra (with 1) of FG. Then, $J(S) \cap FH \leqslant J(S \cap FH)$.

Proof. Choose $x \in J(S) \cap FH$. Then there exists a $y \in J(S)$ such that $x + y + xy = 0$. Let T be any transversal (with 1) to H in G. Suppose $x \neq 0$. Then $y \neq 0$ and we assume that

$$y = \sum_{i=1}^{n} \alpha_i g_i \text{, where } n \geqslant 1,\ 0 \neq \alpha_i \in FH$$

and the g_i distinct elements of T, be the H-decom- -position of y.

$x = -(1 + x)y \in FH$. Thus

$$x = -\sum_{i=1}^{n} (1 + x)\alpha_i g_i \in FH$$ and thus one of

the g_i's, g_1 say, must be 1 and $(1 + x)\alpha_i$ is zero for all $i \neq 1$. But $1 + x$ is a unit of S and hence of FG since x is right-quasi-regular. Thus, $\alpha_i = 0$ for all $i \neq 1$ and thus $y \in FH$.

Let $c \in FH \cap S$, $x \in J(S) \cap FH$. Then by the above xc has a right-quasi-inverse in $S \cap FH$. Thus the right ideal of $S \cap FH$ generated by x is r.q.r. and hence $x \in J(S \cap FH)$ as required.

The main consequence of this lemma is that it enables us to extend results from any class $\mathfrak{X}$ to the class $L\mathfrak{X}$. For example, the following is an immediate consequence of lemmas 3.9 and 3.10.

Corollary 3.11. Let F be a field of characteristic p and G a locally finite group, where if $p \neq 0$, G has no elements of order p. Then FG is Jacobson semi-simple.

We have seen that if $G \in \mathfrak{A}$ we can determine sufficient conditions for a solution. In fact we now show that we are able to obtain sufficient conditions for a solution when $G \in \hat{E}\mathfrak{A}$ and F is a field of characteristic zero. The class $\hat{E}\mathfrak{A}$ is the class that Kurosh called SN in [13] and is the largest class of generalised soluble groups.

The case $G \in \hat{E}\mathfrak{A}$ was first proved by O.E. Villamayor in [26] and later by J.A. Green and S.E. Stonehewer in [8]. We shall outline the proof given in [26] and refer to the proof of Green and Stonehewer in chapter 5. The proof we shall give actually deals with the case when G belongs to the class $\hat{E}\mathfrak{C}$ where $\mathfrak{C}$ is the class of all groups that are locally finite extensions of their centres. Before stating the proof we shall need some local notation.

Let R be a ring and S any subring. Then d(R,S) = 0 if every exact sequence of R-modules that splits as a sequence of S-modules splits also as a sequence of R-modules.

Let G be any group, H a subgroup and T a trans-versal to H in G. We shall say $(G,H)_F \in A$ if whenever $x \in J(FG)$ and $x = \sum_{i=1}^{n} \alpha_i g_i$, $n \geq 1$, is the H-decomposition of x with respect to the transversal T, then the α_i not only belong to FH but also are in fact elements of J(FH) for i = 1, 2, ..., n.

In [26] Villamayor proved some of his results for F, any ring with certain properties. We shall not state the proofs in the same generality which Villamayor used, but will restrict ourselves to the case when F is a field. We begin with the key lemma.

Lemma 3.12. Let R be a ring and P a subring. Suppose that R is a free left-P-module with basis $X = \{u_i\}$ such that $u_i P = P u_i$ and the mapping $\theta_i : p \longrightarrow p_i'$ given by $u_i p = p_i' u_i$ is an automorphism of P. If $\sum_{i=1}^{n} u_i p_i \in J(R)$ and $d(R,P) = 0$, then $p_i \in J(P)$ for $i = 1, 2, \ldots, n$.

The application of this lemma follows from our next lemma due to D.G. Higman. [10].

Lemma 3.13. Let G be a group and H a normal subgroup of finite index, n say. Let F be a field of characteristic p where, if $p \neq 0$, p does not divide n. Then $d(FG,FH) = 0$.

Corollary 3.14. ([26]) Let G be a group and H a normal subgroup, such that $G/H \in L\mathfrak{F}$. Let F be a field of characteristic p where, if p is not zero, G/H has no elements of order p. Then $(G,H)_F \in A$.

Proof. This follows from lemmas 3.10, 3.12 and 3.13. Villamayor's next lemma shows that $(G,H)_F \in A$, when G/H has a series in which each factor belongs to A.

Lemma 3.15. Suppose a class $\mathfrak{X}$ of groups is given such that, if $G/H \in \mathfrak{X}$, where H is a normal subgroup of G, then $(G,H)_F \in A$ for some field F. Let G be a group and N a normal subgroup of G such that G/N has a series with factors belonging to $\mathfrak{X}$. Then $(G,N)_F \in A$.

Corollary 3.16. Let G be a group, H a normal subgroup where $G/H = N$, a free abelian group. Then if F is any field, $(G,H)_F \in A$.

Lemma 3.17. Let G be a group, H a normal subgroup where G/H is locally finite over its centre. Let F be any field of characteristic p where if $p \neq 0$, G/H has no elements of order p. Then $(G,H)_F \in A$.

Proof. Let C be the centre of G/H. Then C has a subgroup M/H such that M/H is free abelian and C/M is locally finite. The result now follows from corollaries 3.14, 3.16 and 3.15.

We define a class $\mathcal{C}$ of groups such that G is a $\mathcal{C}$-group if and only if G is locally finite over its centre. Then clearly $\mathfrak{A} \leqslant \mathcal{C}$.

Theorem 3.18. Let F be a field of characteristic zero and $G \in \hat{E}\mathcal{C}$. Then FG is Jacobson semi--simple.

Theorem 3.19. Let F be any field of characteristic zero. If FS is Jacobson semi-simple for all groups S such that $S = [S,S]$ then FG is Jacobson semi-simple for any group G.

The proof of theorem 3.19 follows from 3.15. Thus, Villamayor has reduced the problem from the class of all groups to the class of all groups that are equal to their own derived groups. However, it appears that he has also removed most of the groups with a sufficient 'nice' structure for a solution to be determined by the present knowledge.

Finally, we shall extend some results of Wallace. In [27] Wallace has shown that the group algebra of a R $\mathfrak{F}$ -group over a field of characteristic p, where if $p \neq 0$, G has no homomorphic images of order divisible by p, is Jacobson semi-simple. We have extended this result to show that if F and G are as above and S is any subalgebra of FG, then $J(S) = \{0\}$. if S is contained in the centre of FG.

We begin with a lemma of our own for the case when G is a finite group.

Lemma 3.20. Let G be a finite group and let F be any field of characteristic p where, if $p \neq 0$, then $p \nmid |G|$. Then, if S is any subalgebra contained in the centre of FG, $J(S) = \{0\}$.

Proof. FG is a finite dimensional vector space over F, thus, S is a finite dimensional vector space over F. Hence, S satisfies the descending chain condition on right ideals. Thus J(S) is nil.

Also, by the remarks at the end of 1.3, $J(S)$ is in fact nilpotent. Assume that $J(S)$ is non zero. Then S contains a non zero ideal C such that $C^2 = \{0\}$. Let $X = \{C + C(FG)\} \neq \{0\}$. Then, X is a non zero two sided ideal of FG and clearly $X^2 = \{0\}$. However, by theorem 2.5, FG has no non zero nilpotent ideals and we get a contradiction. Thus $J(S) = \{0\}$.

We note that if S is any subalgebra that is not contained in the centre, $J(S)$ is not necessarily zero, even when FG is Jacobson semi-simple.
For, let F be the complex field and G be the group $G = \langle a,b\colon a^3 = b^2 = 1,\ a^b = a^{-1} \rangle$. Then FG has a left ideal $A = F(1 + b - a^2b - a) + F(a + ab - b - a^2)$. The ideal $C = F(a + ab - b - a^2)$ is a two sided nilpotent ideal of A.

We now study the class of residually finite groups. We recall that if $G \in R\mathfrak{F}$ then G has a family $\{K_\lambda : \lambda \in \Lambda\}$ of normal subgroups such that G/K_λ is finite for each $\lambda \in \Lambda$ and $\bigcap_{\lambda \in \Lambda} K_\lambda = \langle 1 \rangle$.

In [27] Wallace studied groups having a family $\{H_\lambda : \lambda \in \Lambda\}$ of normal subgroups satisfying the following conditions:

(a) $\bigcap_{\lambda \in \Lambda} H_\lambda = \langle 1 \rangle$;

(b) if $\rho, \sigma \in \Lambda$ then there exists a $t \in \Lambda$ such that $H_t \leq H_\sigma \cap H_\rho$.

If $G \in R\mathfrak{F}$ we can take the family $\{H_\lambda : \lambda \in \Lambda\}$ to be the family $\{K_\lambda : \lambda \in \Lambda\}$ together with the intersections of a finite number of the K_λ's. Clearly this family satisfies (a) and (b).

Let $\mathfrak{X}(G) = \{|G:H_\lambda| : \lambda \in \Lambda\}$ and $\mathcal{Y}(G)$ the set of all prime divisors of the elements of $\mathfrak{X}(G)$. If $G \in \mathfrak{F}_\pi$ for some set of primes π, clearly $\pi = \mathcal{Y}(G)$.

Lemma 3.21. Let G be a group with a family $\{H_\lambda : \lambda \in \Lambda\}$ of normal subgroups satisfying (a) and (b) above. For each $\lambda \in \Lambda$ let I_λ be the kernel of the natural homomorphism from FG to $F(G/H_\lambda)$. Then $\bigcap_{\lambda \in \Lambda} I_\lambda = \{0\}$.

Proof. This is lemma 2.1 of [27].

Theorem 3.22 Let G be a group with a family $\{ H_\lambda : \lambda \in \Lambda \}$ satisfying (a) and (b) above and such that G/H_λ is finite of order n_λ for all $\lambda \in \Lambda$. Let F be a field of characteristic p where, if $0 \neq p$, G/H_λ has no elements of order p. Then:

(i) FG is Jacobson semi-simple.

(ii) if S is any subalgebra (with 1) contained in the centre of FG, then S is Jacobson semi-simple.

Proof. The proof of (i) was given by Wallace in [27].

Proof of (ii). Assume false. Let θ_λ be the natural homomorphism from FG to $F(G/H_\lambda)$. Then the image, $\bar{S}$, of S under θ_λ is a subalgebra contained in the centre of $F(G/H_\lambda)$. Thus, by lemma 3.20, $\bar{S}$ is Jacobson semi-simple. However J(S) is mapped by θ_λ onto a right-quasi-regular ideal of $\bar{S}$. Thus we must have that $J(S) \leq I_\lambda$, the kernel of θ_λ. Thus, $J(S) \leq \bigcap_{\lambda \in \Lambda} I_\lambda = \{0\}$ by lemma 3.21 and we have a contradiction.

Corollary 3.23. Let $G \in R\mathfrak{F}$ and F any field of characteristic p where if $p \neq 0$, $p \notin \mathcal{Y}(G)$. Then FG is Jacobson semi-simple and if S is any subalgebra (with 1) of the centre of FG, $J(S) = \{0\}$.

We recall that Iwasawa has shown that free-groups belong to $R\,\mathfrak{F}_q$ for any prime q. We use this to prove our final corollary.

Corollary 3.24. Let F be any field and G any free group. Then FG is Jacobson semi-simple.

Proof. We choose q to be different to the characteristic of F. Then $G \in R\,\mathfrak{F}_q$ and $\mathfrak{F}_q$ is R_0-closed (i.e. if $G/H \in \mathfrak{F}_q$ and $G/K \in \mathfrak{F}_q$ then $G/H \cap K \in \mathfrak{F}_q$). Thus, FG is Jacobson semi-simple by corollary 3.23.

There are certain other classes of torsion-free groups whose group algebras over any field are Jacobson semi-simple. These will be studied in the next chapter.

Chapter 4. The Jacobson radical of the group algebra of a torsion-free group.

1. In this chapter we are concerned with another of Amitsur's open questions. If F is any field and G any torsion-free group, is FG Jacobson semi-simple? We shall determine for which classes of groups an affirmative answer can be found.

We can see from theorem 3.8 that if G is a torsion-free abelian group and F any field then FG is Jacobson semi-simple. In [25] Stonehewer extended this result to show that FG is Jacobson semi-simple if G is a torsion-free soluble group and F any field. We shall show in section 3, that FG is Jacobson semi-simple for any field F if G belongs to the class of all torsion-free $\acute{E}L\mathfrak{N}$ - -groups. Clearly $\acute{E}\mathfrak{A} \leq \acute{E}L\mathfrak{N}$ and since the former is the class SN*, in the notation of Kurosh [13], our result is true for quite a large class of generalised soluble groups.

The class $\acute{E}L\mathfrak{N}$ was first studied by Plotkin in [22]. In [22] Plotkin has shown that every $\acute{E}L\mathfrak{N}$-group has an ascending normal series with $L\mathfrak{N}$ factors.

We define the following characteristic subgroup of any group G: The Hirsch-Plotkin radical of any group G is the join of all the locally nilpotent normal subgroups of G. Then it is well known that the Hirsch-Plotkin radical is both $L\mathfrak{N}$ and also normal in G. We shall use the notation $\eta(G)$ to denote the Hirsch-Plotkin radical of the group G.

If $G \in \acute{E}L\mathfrak{N}$ we shall choose $\{H_\sigma : \sigma \leq \rho\}$ to be the ascending series for G defined by $H_0 = \langle 1 \rangle$, $H_\rho = G$ and $\frac{H_{\sigma+1}}{H_\sigma} = \eta(G/H_\sigma)$ for all $1 \leq \sigma < \rho$.

In [3] Bovdi has shown that FG is Jacobson semi-simple if G belongs to the class of SN-groups with torsion-free factors. In [14] it is shown that

FG is Jacobson semi-simple for any field F if G belongs to the class of all order preserving trans--formation groups. In the next section we shall extend both these results by showing that FG is Jacobson semi-simple for any field F if G is any $\hat{E}\,\sigma^{*}$ -group where σ^{*} is the class of order preserving transformation groups.

2. On $\mathcal{O}^*$-groups.

We begin with the definition of $\mathcal{O}^*$-groups. Let Λ be any linearly ordered set. Define Orp(Λ) to be the group of all functions f: $\Lambda \longrightarrow \Lambda$ such that f is bijective and if x, y $\in \Lambda$ then the inequality $x < y$ implies that $f(x) < f(y)$. We define $\underline{\mathcal{O}^*}$ to be the class of all groups that can be embedded as a subgroup of Orp(Λ) for some linearly ordered set Λ.

An alternative definition is given by Conrad [5]; $G \in \mathcal{O}^*$ if and only if G can be linearly ordered so that for any a, b, c $\in$ G, $a < c$ implies that $ab < cb$. We define a class $\underline{\mathcal{O}}$ by $G \in \mathcal{O}$ if and only if G can be linearly ordered so that for a, b, c, d $\in G$ and $a < b$, then $cad < cbd$. The class $\mathcal{O}$ is clearly contained in $\mathcal{O}^*$. Also, $\mathcal{O}^*$ is not $\hat{E}$-closed since the (unrestricted) direct product of a family of $\mathcal{O}^*$-groups is not necessary an $\mathcal{O}^*$-group.

Clearly if $G \in \mathcal{O}^*$ then G is torsion-free.

In [16] B. H. Neumann has shown that if G is any torsion-free L$\mathfrak{H}$ -group then G $\in \mathcal{O}$.

Theorem 4.1. Let G be any $\hat{E}\,\mathcal{O}^*$ -group and F any field. Then the following are true:

(i) FG has no non trivial zero divisors.

(ii) if x is a unit of FG, then x has length 1.

(iii) if S is any subalgebra, with 1, of FG, then $J(S) = \{0\}$.

Proof. We shall assume that G has a series $\{\Lambda_\sigma, V_\sigma : \sigma \in \Omega\}$ where for each $\sigma \in \Omega$ $\Lambda_\sigma / V_\sigma$ is an $\mathcal{O}^*$ -group. Let x and y be any two elements of FG. We define s(x,y) = the length of x + the length of y. Clearly $s(x,y) \geq 2$ if $x, y \neq 0$.

We prove (i) by induction on s(x,y). Let x,y be any two elements of FG where xy = 0 and x and y are both non zero.

We assume that if $s(a,b) < s(x,y)$, then ab = 0 implies that a = b = 0. Clearly if s(x,y) = 2, the result is true. Since if $x = \alpha_g g$ and $y = \alpha_h h$

$xy = 0$. Then $\alpha_g \alpha_h = 0$ and since F is a field, we have that α_g or $\alpha_h = 0$.

Assume therefore, that $x, y \neq 0$ and $xy = 0$ where $s(x,y) > 2$. Since if $xy = 0$, then $hxyg = 0$ for all $h, g \in G$, we may assume that 1 belongs to both $\mathrm{Supp}(x)$ and $\mathrm{Supp}(y)$. Now $G \in \hat{E}\theta^{*}$ and this implies that there exists a $\sigma \in \Lambda$ such that $\mathrm{Supp}(x) \cup \mathrm{Supp}(y) \in \Lambda_\sigma = W$, $\mathrm{Supp}(x) \cup \mathrm{Supp}(y)$ does not belong to $V_\sigma = V$. Choose T to be a transversal to V in W.

Let the V-decompositions of x and y be

$$x = \sum_{i=1}^{n} \alpha_i g_i \,, \quad y = \sum_{j=1}^{m} \beta_j h_j, \text{ where } n, m \geq 1,$$

$0 \neq \alpha_i, \beta_j \in FV$ and the g_i's and h_j's elements of T for $1 \leq i \leq n$ and $1 \leq j \leq m$. $m \geq 1$ and $n \geq 1$ Since $x \neq 0 \neq y$, ~~by the choice of $\sigma \in \Lambda$~~ .

Since $W/V \in \mathcal{O}^*$ we can assume that

$g_1 < g_2 < \cdots < g_n$ and $h_1 < h_2 < \cdots < h_m$.

We note that $g_i = h_j = 1$ for some i, j.

Choose l and k such that $g_1 h_l$ is the minimal element of $\{g_1 h_1, g_1 h_2, \ldots, g_1 h_n\}$ and $g_n h_k$ is the maximal element of $\{g_n h_1, g_n h_2, \ldots, g_n h_m\}$.

Then $g_i h_j > g_1 h_l$ if $(i,j) \neq (1,l)$ and $g_i h_j < g_n h_k$ if $(i,j) \neq (n,k)$.

$$\text{Now } xy = 0 = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i (\beta_j)^{g_i^{-1}} g_i h_j.$$

For each pair (i,j) where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, we have $s(\alpha_i, \beta_j) < s(x,y)$. Thus clearly, $s(\alpha_i, (\beta_j)^{g_i^{-1}}) < s(x,y)$. Since α_i and β_j are both non zero, we have by induction that $\alpha_i (\beta_j)^{g_i^{-1}} \neq 0$. Thus in particular

$$\alpha_1 (\beta_l)^{g_1^{-1}} \neq 0 \text{ and } \alpha_n (\beta_k)^{g_n^{-1}}$$

is not zero; we clearly get a contradiction to $xy = 0$.

Proof of (ii). We shall show that if x and y are any two elements of FG such that $xy = g$ for some $g \in G$, then if $1 \in (\mathrm{Supp}(x) \cap \mathrm{Supp}(y))$, both x and y have length equal to 1.

Clearly this implies part (ii). For if $xy = 1$ and $1 \notin (\mathrm{Supp}(x) \cap \mathrm{Supp}(y))$, we can choose g to be any element of $\mathrm{Supp}(x)$ and h to be any element of $\mathrm{Supp}(y)$. Then $(g^{-1}x)(yh^{-1}) = (hg)^{-1}$ and thus $g^{-1}x$ and yh^{-1} are both of length 1. This clearly implies that x and y are both of length 1.

Assume therefore that $xy = g$, $s(x,y) > 2$ and $1 \in (\mathrm{Supp}(x) \cap \mathrm{Supp}(y))$. Since $G \in \hat{E}\mathcal{O}^*$ there exists a unique $\sigma \in \Omega$ such that $\mathrm{Supp}(x) \cup \mathrm{Supp}(y)$ is contained in $\Lambda_\sigma = W$, $\mathrm{Supp}(x) \cup \mathrm{Supp}(y) \nsubseteq V_\sigma = V$. Choose T to be a transversal to V in W.
Let the V-decompositions of x and y be,

$$x = \sum_{i=1}^{n} \alpha_i g_i, \quad y = \sum_{j=1}^{m} \beta_j h_j \text{ , where } n,m \geqslant 1$$

$0 \neq \alpha_i, \beta_j \in FV$ and the g_i's and h_j's are elements of T for $1 \leq i \leq n$, $1 \leq j \leq m$. By the choice of $\sigma \in \Omega$ we can see that $m + n > 2$.

Since $W/V \in \sigma^*$ we can assume that $g_1 < g_2 < \cdots < g_n$ and $h_1 < h_2 < \cdots < h_m$.

We now choose l and k as in the proof of part (i). Thus, $g_1 h_l$ is the minimal element of $\{g_1 h_1, \ldots, g_1 h_m\}$ and $g_n h_k$ is the maximal element of $\{g_n h_1, \ldots, g_n h_m\}$.

Then $g = xy = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i (\beta_j)^{g_i^{-1}} g_i h_j$.

However, for all (i,j) where $1 \leq i \leq n$ and $1 \leq j \leq m$, we have $\alpha_i (\beta_j)^{g_i^{-1}} \neq 0$ by part (i). However, $g_1 h_l < g_i h_j < g_n h_k$ if $(i,j) \neq (1,l)$ or $(i,j) \neq (n,k)$. Thus clearly we must have $m = n = 1$, and we get a contradiction to the choice of $\sigma \in \Omega$.

We have finally to prove part (iii). Let S be any subalgebra with 1 of FG and assume that $J(S) \neq \{0\}$.

If $S = F1$, clearly S is Jacobson semi-simple. Assume therefore that $S \neq F1$. Then there exist a $z \in S$ such that $\mathrm{Supp}(z) \neq \{1\}$. We choose $x \in J(S)$ such that $\mathrm{Supp}(x) \neq \{1\}$. This we can clearly do as $1 \notin J(S)$. $1 + x$ is a unit of S and thus by part (ii) $1 + x$ has length 1. Thus, $x = -1 + \alpha_g g$ for some $g \neq 1$ in G.

Choose an integer n as follows: If the character-istic of F is 0 then $n = 2$, otherwise, if the characteristic of F is $p \neq 0$, n is the least prime such that $(n,p) = 1$. ((n,p) is the greatest common divisor of n and p.)

$x^n \in J(S)$ and thus $x^n = -1 + \alpha_h h$ for some $h \in G$. Thus, the length of x^n is at most 2.

$x^n = (-1 + \alpha_g g)^n$ and since G is torsion-free, 1, g, and g^n are all distinct and occur in the support of x^n. Thus the length of x^n is at least 3 and we get a contradiction. Thus, $J(S) = \{0\}$.

Corollary 4.2. Let $G \in \hat{E}\mathcal{O}^{*}$ and F any field. Then $J(FG) = \{0\}$.

Proof. This follows from setting $S = FG$.

3. Group algebras of torsion-free $\acute{E}L\mathfrak{N}$ -groups.

Throughout this section we will assume that F is any field. We shall prove that the group algebra of a torsion-free $\acute{E}L\mathfrak{N}$ -group over F is Jacobson semi-simple. We begin with a lemma that shows in greater detail the structure of a counter--example, should one exist.

Lemma 4.3. Let G be any group such that the Hirsch-Plotkin radical of G is torsion-free and greater than $\langle 1 \rangle$. Let F be any field such that FG has a non trivial Jacobson radical. Then, there exists a subgroup, G_1, of G such that FG_1 is not Jacobson semi-simple and the centre of G_1 is non trivial.

Proof. Note that we shall not assume that G is torsion-free, only that H, the Hirsch-Plotkin radical of G, is torsion-free.

If G has a non trivial centre then $G = G_1$, will do. If not, choose x to be any non zero element of J(FG). Choose T to be a transversal to H in G. Then x can be written uniquely in the form

$$x = \sum_{i=1}^{n} \alpha_i g_i \text{ , where } n \geq 1,\ 0 \neq \alpha_i \in FH \text{ and}$$

the g_i's distinct elements of T for $1 \leq i \leq n$.

Choose $x \in J(FG)$ of minimal H-length among the non zero elements of J(FG). We can assume that one of the g_i's, g_1 say, is 1 since J(FG) is an ideal of FG.

Each $\alpha_i \in FH$, and thus can be written uniquely in the form $\alpha_i = \sum_{j=1}^{m(i)} \beta_{i,j} h_{i,j}$, where $0 \neq \beta_{i,j} \in F$ and the $h_{i,j}$ belong to H.

Let H(x) be the subgroup

$$H(x) = \langle h_{i,j} : 1 \leq j \leq m(i),\ 1 \leq i \leq n \rangle .$$

We can multiply by $h \in H$ to ensure that $h_{1,1}$ is not 1. Thus we may assume that $H(x) \neq \langle 1 \rangle$.

$H(x)$ is thus a finitely generated group and hence nilpotent. Let $Z \neq \langle 1 \rangle$ be the centre of $H(x)$. Choose h to be any non ~~zero~~ identity element of Z. Then, $x - x^h \in J(FG)$.

$x^h = h^{-1}xh = \sum_{i=1}^{n} \alpha_i [h, g_i^{-1}] g_i$. Thus, clearly

$x - x^h = \sum_{i=2}^{n} \alpha_i (1 - [h, g_i^{-1}]) g_i$. H is a normal subgroup of G and thus $[h, g_i^{-1}] \in H$. Thus, by the choice of x, $x - x^h = 0$. Hence by uniqueness

$\alpha_i (1 - [h, g_i^{-1}]) = 0$ for $2 \leq i \leq n$.

However, H is torsion-free nilpotent and thus $H \in \mathcal{O}$ by [6]. Thus by theorem 4.1 (i), FH has no non trivial zero divisors. We assumed

$\alpha_i \neq 0$ and thus $[h, g_i^{-1}] = 1$ for $2 \leq i \leq n$.

Thus, h commutes with g_i for $1 \leqslant i \leqslant n$.

If we set $G_1 = \langle H(x), g_i : 1 \leqslant i \leqslant n \rangle$,

Then, by lemma 3.10., FG_1 is not semi-simple and clearly Z is contained in the centre of G_1.

Theorem 4.4. Let G be any torsion-free $\acute{E}L\mathfrak{N}$-group and F any field. Then FG is Jacobson semi-simple.

Proof. We assume that the theorem is false. Choose F to be any field for which there exist counterexamples and let $\mathcal{Y}$ be the collection of all $\acute{E}L\mathfrak{N}$-groups with non trivial centres whose group algebra over F is not Jacobson semi-simple. Then $\mathcal{Y} \neq \emptyset$ by lemma 4.3..

We shall in fact find a group G* such that if $H^* = \eta(G^*)$ then H* is the centre of G*. Assuming that we have found G* the rest of the proof proceeds as follows:

Let L/H* be the Hirsch-Plotkin radical of

G^*/H^*. Then L is normal in G^* and L/H^* is locally nilpotent. H^* is the centre of G^* and thus L is locally nilpotent. Hence, $L = H^* = G^*$. Thus, $G^* \in L\mathfrak{N}$ and since G^* is torsion-free $G \in \mathfrak{S}^*$, a contradiction to the ~~second~~ third result of theorem 4.1.

Thus, to prove our result we must find such a G^*. We begin by choosing $G \in \mathcal{Y}$ and let H be the Hirsch-Plotkin radical of G. Choose T to be a transversal to H in G. Then for each non zero element x of $J(FG)$ we will assume that the H-decomposition of x is

$$x = \sum_{i=1}^{n} \alpha_i g_i, \quad 0 \neq \alpha_i \in FH .$$

Let N denote the centre of G. Then $N \leqslant H$ and so we can choose a transversal, S, to N in H. Now for each i, $\alpha_i \neq 0$ and so we can assume

$$\alpha_i = \sum_{j=1}^{m(i)} \alpha_{i,j} h_{i,j} \quad \text{is the N-decomposition}$$

of α_i with respect to S, for $1 \leqslant i \leqslant n$.

Let $s(x) = \sum_{i=1}^{n} m(i)$. Define $r(x)$ to be the ordered pair $(n, s(x))$. Choose $x \in J(FG)$ such that $r(x)$ is minimal with respect to the lexicographic ordering $\leqslant$ on $Z^+ \times Z^+$, among the non zero elements of $J(FG)$.

Then define $r(G) = r(x)$ and choose $G \in \mathcal{Y}$ such that $r(G)$ is minimal with respect to $\leqslant$ among the elements of $\mathcal{Y}$. For this G choose $x \in J(FG)$ such that $r(x) = r(G)$ and, in the H--decomposition of x, $g_1 = h_{1,1} = 1$. This we can clearly do since $r(x) = r(xg)$ for all $g \in G$.

Let $G^* = \langle N, h_{i,j}g_i : 1 \leqslant j \leqslant m(i), 1 \leqslant i \leqslant n \rangle$. Then $N \leqslant N^*$, the centre of G^* and $H \cap G^* \leqslant H^*$, the Hirsch-Plotkin radical of G^*.

Now by lemma 3.10, $x \in J(FG^*)$ and thus $G^* \in \mathcal{Y}$.

Also $h_{i,j}g_i$ and $h_{i,1}g_i$ belong to the same coset of H^* in G^*. Let this coset be $H^*g^*_i$, where g^*_i belongs to some transversal T^* to H^* in G^*, for $1 \leq i \leq n$. Also $g^*_1 = 1$.

Thus, $h_{i,j}g_i$ can be written uniquely in the form $h_{i,j}g_i = n^*_{i,j}h^*_{i,j}g^*_i$, where $n^*_{i,j} \in N^*$ and $h^*_{i,j}$ belongs to a transversal S^* to N^* in H^*.

W.L.O.G. assume that $h^*_{1,1} = 1$. Then x can be written in the form
$$x = \sum_{i=1}^{n} \sum_{j=1}^{m(i)} \beta_{i,j}n^*_{i,j}h^*_{i,j}g^*_i.$$

Thus by the choice of G and n, the g^*_i's are distinct and by the choice of x, for fixed i, the $h^*_{i,j}$'s are distinct. We set $\beta^*_{i,j} = \beta_{i,j}n^*_{i,j}$.

Let B denote the subgroup

$$\langle N^*, h^*_{i,j} : 1 \leqslant j \leqslant m(i),\ 1 \leqslant i \leqslant n \rangle.$$

Then $B \leqslant H^*$. Also, B/N^* is finitely generated and thus B/N^* is nilpotent. Clearly, since N^* is the centre of G^*, B is nilpotent. Also, $N^* \leqslant M$, where M is the centre of B.

Let $\alpha^*_i = \sum_{j=1}^{m(i)} \beta^*_{i,j} h^*_{i,j}$. Then choose $g \in M$. Since $J(FG^*)$ is an ideal of FG^*, $x - x^g$ belongs to $J(FG^*)$. However,

$$x - x^g = \sum_{i=2}^{n} \alpha^*_i (1 - [g, g^{*\,-1}_i]) g^*_i .$$

$[g, g^{*\,-1}_i] \in FH^*$ for $1 \leqslant i \leqslant n$ since H^* is normal in G^* and thus $x - x^g = 0$ by the choice of x minimal. Thus, by uniqueness $\alpha^*_i (1 - [g, g^{*\,-1}_i]) = 0$ for $2 \leqslant i \leqslant n$. Hence, by ~~lemma~~ Theorem 4.1 (i) we must have $[g, g^{*\,-1}_i] = 1$ and thus $g \in N^*$. Thus, $N^* = M$.

Let K/M be the centre of B/M. Then choose k to be any non ~~zero~~ identity element of K.

Then $x^k = \alpha_{1,1} + \sum_{j=2}^{m(1)} \beta^*_{1,j}[k,h^*_{1,j}{}^{-1}]h^*_{1,j} +$

$$+ \sum_{i=2}^{n} (\alpha^*_i)^k[k,g^*_i{}^{-1}]g^*_i.$$

Also $[k,h^*_{1,j}{}^{-1}] \in M$ for all $2 \leqslant j \leqslant m(1)$ since $\frac{K}{M}$ is the centre of $\frac{B}{M}$ and $[k,g^*_i{}^{-1}] \in H^*$ for $2 \leqslant i \leqslant n$ since H^* is normal in G^*. Clearly $x - x^k \in J(FG^*)$

and $x - x^k = \sum_{j=2}^{m(1)} \beta^*_{1,j}(1 - [k,h^*_{1,j}{}^{-1}])h^*_{1,j} +$

$$+ \sum_{i=2}^{n} (\alpha^*_i - (\alpha^*_i)^k[k,g^*_i{}^{-1}])g^*_i$$

$= 0$ by the choice of x.

Thus, $\beta^*_{1,j}(1 - [k,h^*_{1,j}{}^{-1}]) = 0$ and

$\alpha^*_i - (\alpha^*_i)^k[k,g^*_i{}^{-1}] = 0$ for $1 \leqslant j \leqslant m(1)$, $1 \leqslant i \leqslant n$.

Hence $\alpha^*_i g^*_i = (\alpha^*_i g^*_i)^k$ for $1 \leqslant i \leqslant n$ and all $k \in K$.

Hence the number of conjugates of each $h^*_{i,j}g^*_i$ under transformation by the elements of K cannot exceed $|\mathrm{Supp}(\alpha^*_i)|$. Thus $K/(K \cap N^*) = K/M$ is finite. However, B is nilpotent torsion-free and thus by [15] each factor of the upper central series of B is torsion-free. Hence K/M is also torsion-free and thus $K = M = B$.

Thus each of the α^*_i's belongs to FN^*. Choose $h \in H^*$. Then $x - x^h \in J(FG^*)$.

$$x^h = \sum_{i=1}^{n} \alpha^*_i[h,g^*_i{}^{-1}]\, g^*_i \quad \text{, where } [h,g^*_i{}^{-1}] \in H^*.$$

$$x - x^h = \sum_{i=2}^{n} \alpha^*_i \,(1 - [h,g^*_i{}^{-1}])g^*_i = 0,$$

by the choice of x. Hence for $i = 2, 3, \ldots, n$

$\alpha^*_i \,(1 - [h,g^*_i{}^{-1}]) = 0$. Theorem 4.1. (i) then implies that since $\alpha^*_i \neq 0$, we must have that $[h,g^*_i{}^{-1}] = 1$, for $2 \leqslant i \leqslant n$. Hence $h \in N^*$ and thus $N^* = H^*$ and we have found the required G^*.

Chapter 5. Twisted group algebras.

1. In this chapter we introduce another ~~a~~ associative algebra which can be formed from a group and a field. In fact it is a generalisation of the usual group algebra defined in chapter 1.

Let F be a field and G a group. We let F^tG denote the <u>twisted group algebra of G over F.</u> That is, F^tG is an associative algebra over F with basis $\{\bar{x} : x \in G\}$ and with multiplication defined by

(i) $\bar{x}\bar{y} = \gamma(x,y)\overline{xy}$, where $\gamma(x,y)$ belongs to $F^* = F - \{0\}$.

The associative condition is equivalent to

(ii) $\gamma(x,yz)\,\gamma(y,z) = \gamma(x,y)\,\gamma(xy,z)$.

If all $\gamma(x,y) = 1$ then F^tG is in fact FG.

In the main part of this chapter we will study the radicals of certain twisted group algebras.

We will in fact, prove the obvious analogues of the results in chapters 2 and 3.

Before we commence our study, however, we note the following lemma concerned with the structure of F^tG. This was first proved in [20].

Lemma 5.1. Let F be any field and G any group. Let x be any element of G. Then:

(i) $\quad 1 = \gamma(1,1)^{-1}\overline{1}$.

(ii) $\quad \overline{x}^{-1} = \gamma(1,1)^{-1} \quad \gamma(x,x^{-1})^{-1}\overline{x^{-1}} =$

$$= \gamma(1,1)^{-1} \quad \gamma(x^{-1},x)^{-1}\overline{x^{-1}}.$$

Thus, we can see that F^tG is an associative algebra with 1.

2. The nil radical and nilpotent ideals.

The results of this section are taken from D. S. Passman [20]. We begin with a result on the nil radical.

Theorem 5.2. Let G be any group. Then if F is any field of characteristic p such that F is algebraically closed, $F^t G$ has a non zero nil radical implies that $p > 0$ and G has an element of order p,

Thus, theorem 5.2 gives sufficient conditions for a twisted group algebra to be nil semi-simple. As in chapter 2, we are unable to determine necessary conditions.

Before turning our attention to nilpotent ideals we must define the following characteristic subsets of a group G: $\Delta_0(G) = \{g \in G : |G: \mathcal{C}_G(g)| \text{ is finite}\}$.

$$\Delta_1(G) = \{g \in \Delta_0(G) : g \text{ has finite order}\}.$$

$$\Delta_p(G) = \{g \in \Delta_1(G) : g \text{ has order a power of } p\}.$$

~~Then it follows from Dietzmann's lemma that~~ ~~$\Delta_0(G)$ and $\Delta_1(G)$ are characteristic subgroups of G.~~

Our final result in this section is the main result of [20]. We assume F is algebraically closed.

Theorem 5.3. Let G be a group, F a field of characteristic p and F^tG, the twisted group algebra. Then if F is of characteristic zero $W(F^tG) = \{0\}$. If F is of characteristic $p > 0$ then the following hold:

(a) $W(F^tG) = W(F^t(\Delta_p(G)))\, F^tG$.

(b) $W(F^t(\Delta_p(G))) = J(F^t(\Delta_p(G))) = \bigcup J(F^tM)$,

where the union is over all finite normal subgroups M of G with $M \leq \Delta_p(G)$.

(c) $W(F^tG) = \{0\}$ if and only if $\Delta_p(G) = \langle 1 \rangle$.

Thus, as in chapter 2, we can determine necessary and sufficient conditions for F^tG to have no non zero nilpotent ideals.

3. Subgroups.

In this section we shall determine the relation--ship between $J(F^tG)$ and $J(F^tH)$ where H is a normal subgroup of G and G/H is either abelian or finite. We shall assume that F is an algebraically closed field of characteristic p. The source for this section is again [20].

Lemma 5.4. Let H be a normal subgroup of G of index n. Then

$$(J(F^tG))^n \leqslant J(F^tH)(F^tG) \leqslant J(F^tG).$$

Lemma 5.5. Let H be a normal subgroup of G of index n. Assume that n and p are coprime. Then,

$$J(F^tG) = J(F^tH)(F^tG).$$

If H is any subgroup of G we are able to prove the following analogue of lemma 3.10.

Lemma 5.6. Let H be a subgroup of G. Then

$$J(F^tG) \cap F^tH \leqslant J(F^tH).$$

We are also able to prove the following:

Theorem 5.7. Let G be a group, H a normal sub-group such that G/H is abelian and assume that if $p \neq 0$. G/H has no elements of order p. If I is any characteristic ideal of F^tG, then

$$I = (I \cap F^tH)(F^tG).$$

In the next chapter we shall study the Jacobson radical of F^tG, where G belongs to some class $\mathfrak{X}$. We shall show, for certain conditions on the field F, that F^tG is Jacobson semi-simple if $\mathfrak{X}$ is the class $\mathfrak{F}$, $E\mathfrak{A}$ or $\hat{E}\mathfrak{A}$.

We recall that in chapter 3 it was shown that FG is Jacobson semi-simple for certain F, if $G \in R\mathfrak{F}$. We have been unable to prove this result for twisted group algebras. The reason why the analogue of theorem 3.22 does not work will be shown in the next section.

4. The Jacobson radical.

In this section we shall study the Jacobson radical of F^tG. Our first result enables us to apply Theorem 3.7. Theorem 3.7, we recall, states that if A is a nilpotent-free algebra over F, and K the algebraic closure of F, $J(A) = \{0\}$ if and only if $J(A_K) = \{0\}$.

Theorem 5.8. Let G be any group and F any field of characteristic p. If $p = 0$ F^tG is nilpotent-free. If $p \neq 0$, F^tG is nilpotent-free if and only if $\Delta_p(G) = \langle 1 \rangle$.

This is proposition 2.1 of [21].

Thus, we can clearly see that the proofs of our first two results follow from section 3 by an application of theorems 3.7 and 5.8.

Theorem 5.9. ([21]) Let F be a field of character--istic p and $G \in L\mathfrak{F}$. Then if $p \neq 0$, F^tG is Jacobson semi-simple if and only if G has no element of order p. If $p = 0$, $J(F^tG) = \{0\}$.

The proof follows by putting $H = \langle 1 \rangle$ in lemma 5.5. We then use theorems 5.8, 3.7 and lemma 5.6 to obtain the proof.

Before our next theorem, which is concerned with the class $E\mathfrak{O}\mathfrak{l}$, we shall need to prove a lemma that will enable us to use induction.

Lemma 5.10. ([21]) Let H be a normal subgroup of G with $G/H \in \mathfrak{O}\mathfrak{l}$. Suppose that $J(F^tH) = \{0\}$ for some field F of characteristic p, where, if p is not zero, G has no elements of order p. Then $J(F^tG) = \{0\}$.

The proof is given by Passman in [21]. Passman then used this lemma and induction on the derived length of the group G, to prove the following theorem.

Theorem 5.11. ([21]) Let $G \in E\mathfrak{O}\mathfrak{l}$ and F any field of characteristic p where, if $p \neq 0$, G has no element of order p. Then F^tG is Jacobson semi-simple.

Let G be a group and H a normal subgroup. Then the natural homomorphism from G to G/H induces a group algebra homomorphism from FG to $F(G/H)$. However, there is no similar twisted group algebra homomorphism from F^tG to $F^t(G/H)$ unless $H = \langle 1 \rangle$ or $FG = F^tG$. This follows from the fact that the coset representative chosen determines the field element introduced into the multiplication. Thus, if we attempt to copy the proof of theorem 3.18, we will be unable to prove the analogue of theorem 3.15 for F^tG. However, we are still able to determine sufficient conditions for F^tG to be Jacobson semi-simple when $G \in \hat{E}\mathfrak{A}$. We do this by a method which is similar to that used in [8] by Green and Stonehewer.

If we attempt to prove the analogue of theorem 3.22 we are unable to copy the proof given there. However, we have so far been unable to prove the analogue of lemma 3.21.

Theorem 5.12. Let F be any field of characteristic zero and $G \in \hat{E}\mathfrak{A}$. Then F^tG is Jacobson semi-simple.

Proof. By theorems 5.8 and 3.7 we can assume that F is algebraically closed, since F^tG is nilpotent free. Assume that F^tG is not Jacobson semi-simple and choose x to be any non zero element of $J(F^tG)$. Then x can be written uniquely in the form

$$x = \sum_{i=1}^{n} \lambda_i \bar{g}_i, \text{ where } n \geqslant 1,\ 0 \neq \lambda_i \in F \text{ and}$$

the g_i's are distinct elements of G, for $1 \leqslant i \leqslant n$.

Since $J(F^tG)$ is an ideal, we can set x' to be $x\lambda_1^{-1}\bar{g}_1^{-1}$. Then $x' \in J(F^tG)$ and $g_1 = 1$. Also, $\lambda_1\bar{g}_1$ is the unit of F^tG. $\bar{g}_1^{-1}$ exists by lemma 5.1.

Let $H = \langle \mathrm{Supp}(x') \rangle$. Then $H \in \hat{E}\mathfrak{A} \cap \mathfrak{G}$.

Thus, H has a series $\{\Lambda_\sigma, V_\sigma : \sigma \in \Omega\}$ such that $\Lambda_\sigma / V_\sigma$ is cyclic of prime order. Since H is finitely generated, $H = \Lambda_\omega$ for some $\omega \in \Omega$.

Let $N = V_{\omega}$. Then $H = \langle N, t \rangle$ where $t^p \in N$.

Now each $0 \neq x \in F^tH$ can be written uniquely in the form

$$x = \sum_{i=0}^{p-1} \theta_i \overline{t^i}, \quad \theta_i \in F^tN.$$

Let μ be a primitive pth root of 1 in F.

Then ϕ defined by

$$\phi : \sum_{i=0}^{p-1} \theta_i \overline{t^i} \longmapsto \sum_{i=0}^{p-1} \theta_i \overline{t^i} \mu^i$$

is an automorphism of F^tH.

Let $g_i = n_i t^{m_i}$, $n_i \in N$, $0 \leqslant m_i \leqslant p-1$.

Then since $g_1 = 1$, $n_1 = 1$, $m_1 = 0$.

$$x' = \sum_{i=1}^{n} \theta_i \gamma(n_i, t^{m_i})^{-1} \overline{n_i} \overline{t^{m_i}}.$$

Since $H = \langle \mathrm{Supp}(x') \rangle$, there exists $m_j \neq 0$, say $m_2 \neq 0$. Then by lemma 5.6, $x' \in J(F^tH)$ and since $J(F^tH)$ is an invarient ideal of F^tH, $x\phi - \mu^{m_2}x$ belongs to $J(F^tG)$.

$$x\phi = \sum_{i=1}^{n} \theta_i \mu^{m_i} \gamma(n_i, t^{m_i})^{-1} \overline{n_i} \overline{t^{m_i}}.$$

$$\mu^{m_2} x = \sum_{i=1}^{n} \mu^{m_2} \lambda_i \gamma(n_i, t^{m_i})^{-1} \overline{n_i t^{m_i}}.$$

Thus, $y = x\phi - \mu^{m_2}x \in J(F^tH)$ and has smaller length than x. Also y is non zero. It follows that there exists a $\hat{E}\mathfrak{A}$-group M, where M is a subgroup of G, and an element of F^tM of length 1, which clearly leads to a contradiction.

Corollary 5.12. Let F be any field of characteristic zero and G a finite extension of a $\hat{E}\mathfrak{A}$-group. Then F^tG is Jacobson semi-simple.

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Chapter 6. The Brown-McCoy radical.

In this chapter we are concerned with the Brown-McCoy radical of FG . We recall that the Brown-McCoy radical of a ring R is the maximal B-regular ideal of R. $a \in R$ is defined to be B-regular if the two sided ideal, B(a) is equal to the whole ring R. Note that if R has a unit $B(a) = \left\{ \sum_i x_i (a + 1) y_i : x_i , y_i \in R \text{ and sum finite} \right\}$

It is clear that B-regularity is the most difficult of the three radical properties that we are studying. In fact, nearly all our results depend on showing that FG is Brown-McCoy semi--simple if and only if S is Jacobson semi-simple for some subalgebra , S, contained in the centre of FG. We are then able to determine sufficient conditions for J(S) to be zero provided that G belongs to a certain class of groups.

We begin by studying the class ZA. G is defined to be a ZA-group if and only if G has an ascending central series. We are able to show that if $G \in ZA$ and F is a field of characteristic p, FG is Brown-McCoy semi-simple if G is torsion-free or if p does not belong to a certain set of primes, determined by G.

$ZA \leq \acute{E}\mathfrak{A}$. However, we have been unable to extend our results to the larger class. In fact, we are still unable to obtain a complete solution when G is any metabelian group.

The main reason for this difficulty in extending our result is that the proof of one of our key lemmas, Lemma 6.1, depends on the ascending central series in G being central. Further, we can determine no apparent reason why $G \in \acute{E}\mathfrak{A}$ implies the centre of FG is greater than F1. We have however, been ~~un~~able to prove that FG is Brown-McCoy semi-simple for some special types of metabelian groups.

We are also able to show that FG is Brown--McCoy semi-simple if G is any $R\mathfrak{F}$ -group and F a field of characteristic p, where p is determined by G.

We recall that if G is a soluble group with the maximal condition then G is polycyclic and hence $R\mathfrak{J}$. If we study soluble groups with the minimal condition however, we have been unable to obtain a solution. We recall that a soluble group with the minimal condition is a finite extension of a divisible abelian group. We are able to show that if it is a cyclic prime extension of a divisible abelian group then it has a Brown-McCoy semi-simple algebra over a field of suitably chosen characteristic.

2. On ZA-groups.

We begin with two key lemmas.

Lemma 6.1. Let G be any ZA-group and F any field. Assume that $\{V_\alpha : \alpha \leqslant \rho\}$ is any ascending central series of G. Let S (> 0) be a vector subspace of FV_α for some $\alpha \leqslant \rho$, satisfying,

(i) if $s \in S$ and $g \in G$, then $g^{-1}sg \in S$ and

(ii) if $s \in S$ and $h \in V_\alpha$, then $sh \in S$.

Then $S \cap Z \neq \{0\}$, where Z is the centre of FG.

Proof. Assume that the lemma is false. Let α be the first ordinal for which there exists such an $S \leqslant FV_\alpha$ with $S \cap Z = \{0\}$. Then if $\beta < \alpha$, $S \cap FV_\beta = \{0\}$. For if not, $S^* = S \cap FV_\beta$ is a vector subspace of FV_β satisfying (i) and (ii) above and $S^* \cap Z = S \cap FV_\beta \cap Z = \{0\}$, contradicting the choice of α. Thus, clearly α is not a limit ordinal and thus α is equal to $\gamma + 1$, for $\gamma > 1$.

Choose x to be any non zero element of S. Then x can be written uniquely in the form

$$x = \sum_{i=1}^{n} \theta_i g_i, \; n \geqslant 1, \; 0 \neq \theta_i \in FV_\gamma \text{ and the}$$

g_i's belonging to some transversal T_1 to V_γ in $V_{\gamma+1}$. Choose x from among the non zero elements of S such that n, the V_γ-length of x, is minimal. We can assume that $g_1 = 1$, since if not $xg_1^{-1} \in S$ by condition (ii).

We now fix $g_1, g_2, \ldots, g_n$ as they occur in the H-decomposition of x. We define a set X by $X = \{y : y = \phi_1 g_1 + \phi_2 g_2 + \cdots + \phi_n g_n \in S$ and $\phi_i \in FV_\gamma$ for $1 \leqslant i \leqslant n\}$.

Then $X \neq \{0\}$ since $x \in X$. Let S_1 be a subset of FV_γ defined by $S_1 = \{\phi_1 :$ there exist $\phi_2, \phi_3, \ldots, \phi_n \in FV_\gamma$ and $y = \phi_1 g_1 + \phi_2 g_2 + \cdots + \phi_n g_n$ belongs to $X.\}$

Then $S_1 > \{0\}$, since $\Theta_1 \in S_1$. We now claim that S_1 is a vector subspace of FV_γ satis--fying (i) and (ii).

Let ϕ_1 and ϕ_1' be any two elements of of S_1. Then there exists an x and an x' belonging to X such that

$x = \phi_1 g_1 + \phi_2 g_2 + \cdots + \phi_n g_n$, and

$x' = \phi_1' g_1 + \phi_2' g_2 + \cdots + \phi_n' g_n$, for some

$$\phi_2, \phi_3, \ldots, \phi_n, \phi_2', \phi_3', \ldots, \phi_n'$$

belonging to FV_γ .

Then $x \pm x' \in S$ and

$$x \pm x' = (\phi_1 \pm \phi_1')g_1 + (\phi_2 \pm \phi_2')g_2 + \cdots + (\phi_n \pm \phi_n')g_n.$$

Thus $x + x' \in X$ and hence $\phi_1 \pm \phi_1' \in S_1$.

We now show that S_1 satisfies (i). Choose

$\phi_1 \in S_1$ and $g \in G$. Then there exist ϕ_2, $\phi_3, \ldots, \phi_n$ belonging to FV_γ and $x \in X$ such that

$$x = \phi_1 g_1 + \phi_2 g_2 + \cdots + \phi_n g_n \in S.$$

$$x^g = g^{-1}xg = (\phi_1)^g + (\phi_2)^g[g, g_2^{-1}]\, g_2 + \cdots + (\phi_n)^g[g, g_n^{-1}]\, g_n \in S, \text{ since } S$$

satisfies (i).

However, $[g, g_i^{-1}] \in V_\gamma$ for $2 \leqslant i \leqslant n$, since $\{V_\alpha : \alpha \leqslant \rho\}$ is a cetral series of G. Thus, $x^g \in X$ and consequently $(\phi_1)^g \in S_1$ and S_1 satisfies (i).

Thus it is clear that S_1 is a vector subspace satisfying (i). We now show that it satisfies (ii) also.

Choose $\phi_1 \in S_1$ and $h \in V_\gamma$. Then there exist $\phi_2, \phi_3, \ldots, \phi_n \in FV_\gamma$ and $y \in X$

such that $y = \phi_1 g_1 + \phi_2 g_2 + \cdots + \phi_n g_n \in S$.

Then $yh \in S$, since S satisfies (ii) and

$$yh = \phi_1 h g_1 + \phi_2 (h)^{g_2^{-1}} g_2 + \cdots + \phi_n (h)^{g_n^{-1}} g_n.$$

Since V_γ is normal in G for each $\gamma \leqslant \rho$, $h^{g_i^{-1}} \in V_\gamma$ and thus $\phi_i h^{g_i^{-1}} \in FV_\gamma$ for all $2 \leqslant i \leqslant n$. Thus, $xh \in X$ and $\phi_1 h \in S_1$ as required.

Now by the choice of α we must have that $S_1 \cap Z \neq 0$. We then choose θ_1 to be any non zero element of this intersection. Then there exist $\theta_2, \theta_3, \ldots, \theta_n \in FV_\gamma$ and a $y \in X$ such that

$$y = \theta_1 g_1 + \theta_2 g_2 + \cdots + \theta_n g_n \in S.$$

However, $y - y^g \in S$ for all $g \in G$ and

$$y - y^g = \sum_{i=2}^{n} (\theta_i - (\theta_i)^{g[g, g_i^{-1}]}) g_i.$$

Clearly, $(\theta_i - (\theta_i)^g[g,g_i^{-1}]) \in FV\gamma$ and thus by the choice of n minimal, we must have $y - y^g = 0$ for all $g \in G$. Thus $y \in S \cap Z$ and we have a contradiction. Hence, $S \cap Z \neq \{0\}$ as required.

Lemma 6.2. Let R be a ring with 1, B the Brown-McCoy radical of R and Z the centre of R. Then, $B \cap Z \leqslant J(Z)$, where J(Z) denotes the Jacobson radical of Z.

Proof. If $B \cap Z = \{0\}$ the lemma is trivial. Assume, therefore, that $B \cap Z \neq \{0\}$. Choose x to be any non zero element of the intersection. We aim to show that the right ideal of Z generated by x is right-quasi-regular. Let $a = xz$ for $z \in Z$.

$-a \in R$ and since $a \in B$, for some $x_i, y_i \in R$ and some integer $n > 0$,

$-a = \sum_{i=1}^{n} x_i (a + 1)y_i$. Thus, $a = - \sum_{i=1}^{n} (a + 1)x_i y_i$

$$= - (a + 1) \sum_{i=1}^{n} x_i y_i = - (a + 1) r, \text{ where}$$

$$r = \sum_{i=1}^{n} x_i y_i .$$

Then $a + ar + r = 0$ for $r \in R$ and so a is right-quasi-regular in R. We now show that $r \in Z$.

Choose y to be any element of R. Then,

$y(a + r + ar) - (a + ar + r)y = 0.$

i.e. $yr - ry + a (yr - ry) = 0$ since $a \in Z$

Thus $u = yr - ry = -au = (r + ar)u$

$= (r + ra)u = ru + (-ru) = 0.$

Thus, y commutes with r and thus $r \in Z$. But $a = xz$ and z was chosen arbitrarily in Z and thus xZ is a right-quasi-regular right ideal of Z and thus contained in $J(Z)$.

In the next section we shall prove our main two theorems. It is in the proof of this theorem that we use lemma 3.10 in the generality in which we proved it. We also use certain properties of $\mathcal{G} \cap \mathcal{H}$ -groups, with which we begin the section.

Lemma 3.10 implies that in order to determine the Jacobson semi-simplicity of Z, we need only consider sufficient conditions for $Z \cap FH$ to be Jacobson semi-simple for any $\mathcal{G}$-subgroup H of G. Clearly if $G \in ZA$, then H is a $\mathcal{G} \cap \mathcal{H}$-group.

Theorem 6.3. Let G be a polycyclic group. Then, G is an $R\mathcal{F}_{\pi}$-group for some finite set of primes π. Further, if G is torsion-free, π can be chosen to be any prime p.

Proof. This follows directly from theorems E and F of [24].

The set π is constructed in the proof of this theorem and, as Robinson has pointed out, there exists a certain degree of choice as to which primes occur. Let $\omega(G) = \{\pi :$ such that G is an $R\mathcal{F}_{\pi}$ -group$\}$.

Let $\mathcal{H}(G) = \bigcap_{\pi \in \omega(G)} \pi$.

Let G be a group and $H \in \mathcal{G} \cap \mathcal{N}$ a normal subgroup of G. We define a set of primes A(G) by A(G) = any prime p if G is torsion-free, and A(G) is equal to $\bigcup_{H} \varpi(H)$ where the intersection ranges over all normal subgroups H such that $H \in \mathcal{G} \cap \mathcal{N}$.

Then A(G) is a (possibly infinite) set of primes. We now prove the main result of this chapter.

Theorem 6.4. Let G be a ZA-group and F any field of characteristic p where, if $p \neq 0$, $p \notin A(G)$. Then FG is Brown-McCoy semi-simple.

Proof. Assume not. Let $B \neq \{0\}$ denote the Brown--McCoy radical of FG. Then by lemma 6.1, $B \cap Z \neq \{0\}$ where Z denotes the centre of FG. Lemma 6.2 then gives $J(Z) \neq \{0\}$. Choose x to be any non zero element of J(Z) and let H be the subgroup of G generated by the support of x. Since $x \in Z$, $x = x^g$ for all $g \in G$. Then H is a normal subgroup of G. $G \in ZA$ and thus $G \in L\mathcal{N}$, [14]. Then $H \in \mathcal{G} \cap \mathcal{N}$, By the well used lemma 3.10, $J(Z \cap FH) \neq \{0\}$. However, $Z \cap FH$ is a subalgebra (with 1) contained in the centre of FH. $H \in R\mathfrak{F}_{\pi}$ for some finite set of primes π such that $p \notin \pi$. Since F is a field of characteristic p, $J(Z \cap FH) = \{0\}$ by lemma 3.23. Thus we have a contradiction.

Theorem 6.5. Let F be any field and G any torsion--free ZA-group. Then FG is Brown-McCoy semi-simple.

Proof. By definition we can choose A(G) to be any prime p. Let A(G) be the least prime different to the characteristic of F. Then the result follows from theorem 6.4.

Alternatively we have that lemmas 6.1 and 6.2 imply that $J(Z) \neq \{0\}$ and since $ZA \leq L\mathfrak{N}$ we can use theorem 4.1 (iii) to obtain a contradiction. Since if G is a torsion-free $L\mathfrak{N}$ -group then G belongs to $\mathcal{O}^*$ by [16].

We are now able to prove the following result in the case when G is periodic.

Theorem 6.6. Let G be a periodic ZA-group and F a field of characteristic p, where if $p \neq 0$, G has no finite normal subgroup whose order is divisible by p. Then FG is Brown-McCoy semi-simple.

Proof. Assume not. Let $B \neq \{0\}$ denote the Brown-McCoy radical. Then, as in the proof of theorem 6.4, we get that $J(Z) \neq \{0\}$, where Z is the centre of FG. Let H be the subgroup of G generated by the support of x where x is any non zero element of J(Z). $x = x^g$ for all $g \in G$ and thus H is a normal subgroup of G. $H \in \mathcal{G} \cap \mathcal{H}$ -group and thus H is finite. By lemma 3.10, $x \in J(Z \cap FH)$ and thus $Z \cap FH$ is not Jacobson semi-simple. However, lemma 3.20 now gives a contradiction.

3. On metabelian groups.

We begin with the following technical lemma:

Theorem ~~Lemma~~ 6.7. Let G be any abelian group and F any field of characteristic p where, if $p \neq 0$, G has no elements of order p. Then if S is any sub-algebra with 1, of FG, $J(S) = \{0\}$.

Proof. If H is a subgroup of G it is clear from Theorem ~~lemma~~ 3.7 and the remarks following Theorem ~~lemma~~ 3.8, that $J(FH) = \{0\}$.

Assume that J(S) is non zero and choose x to be any non zero element of J(S). Let H be the subgroup of G generated by the support of x. Then since Supp(x) is a finite set, H is finitely generated and thus $H = A \times B$, where A is finite and B is free abelian.

Let π be the set of all prime divisors of the order of A. Then $B \in R\mathfrak{F}_{\pi}$ and thus B has a family $\{H_\sigma : \sigma \in \Omega\}$ such that $\bigcap_{\sigma \in \Omega} H_\sigma = \langle 1 \rangle$ and B/H_σ is a finite π-group. Each H_σ is in fact normal in H and thus H has a family of normal subgroups, $\{H_\sigma : \sigma \in \Omega\}$ such that H/H_σ is a finite π-group and they ~~satisfies~~ satisfy conditions (a) and (b) of ~~lemma~~ Theorem 3.22. By the initial conditions, $p \notin \pi$ and thus by lemma 3.22 we have $J(S \cap FH)$ is zero. Lemma 3.10 gives a contradiction.

The main application of this lemma is the following result:

Lemma 6.8. Let G be a group and H an abelian normal subgroup of finite index in G. Let F be a field of characteristic p where, if $p \neq 0$, G and G/H have no elements of order p. Then if B denotes the Brown-McCoy radical of FG, $B \cap FH = \{0\}$.

Proof. Assume not and let $x \neq 0$ denote any element of the intersection. We can write x uniquely in the

form $x = \sum_{i=1}^{n} \lambda_i g_i$, $n \geqslant 1$, $0 \neq \lambda_i \in F$ and

the g_i's distinct elements of H. Since $B \cap FH$ is an ideal of FH we can assume that one of the g_i's, g_1 say, is 1. Let $h_1, h_2, \ldots, h_m$ be a complete set of coset representatives to H in G. Then,

$$x^{h_j} = \lambda_1 1 + \sum_{i=2}^{n} \lambda_i (g_i)^{h_j}, \; 1 \leqslant j \leqslant m.$$

Let $y = \sum_{j=1}^{m} x^{h_j}$. Then $y = \sum_{j=1}^{m} \sum_{i=1}^{n} \lambda_i (g_i)^{h_j}$

$$= m \lambda_1 1 + \sum_{j=1}^{m} \sum_{i=2}^{n} \lambda_i (g_i)^{h_j}.$$

Then y is a non zero element of the centre of FG, by the choice of F. $y \in B \cap FH \cap Z$, where Z is the centre of FG. Thus by lemmas 6.2 and 3.10 $y \in FH \cap J(Z) \leqslant J(Z \cap FH)$.

$Z \cap FH$ is a subalgebra with 1 of FH and thus by lemma 6.7 $J(Z \cap FH) = \{0\}$. This gives a contra--diction and thus $B \cap FH = \{0\}$.

Thus we have seen that if H is a subgroup of finite index in G such that H is abelian, then $B \cap FH = \{0\}$. We shall use this result to show that for certain conditions on F and G, $B(FG) = \{0\}$.

Lemma 6.9. Let G be a group, H, a normal subgroup such that H is a divisible abelian group and G/H is cyclic of prime order. Let F be a field of characteristic p where, if $p \neq 0$, G and G/H have no elements of order p. Then FG is Brown-McCoy semi-simple.

Proof. Assume that the lemma is false and let $B \neq \{0\}$ denote the Brown-McCoy radical of FG. Choose x to be any non zero element of B and let T be a transversal to H in G such that T contains 1.
Let the H-decomposition of x be

$$x = \sum_{i=1}^{n} \alpha_i g_i \ , \ n \geqslant 1, \ 0 \neq \alpha_i \in FH \text{ and the}$$

g_i's distinct elements of T for $1 \leqslant i \leqslant n$.

Choose x from among the elements of B such that x has minimal H-length. W.L.O.G. we can assume that one of the g_i's, g_1 say is 1.

Let $h \in H$. Then $x^h = \sum_{i=1}^{n} \alpha_i [h, g_i^{-1}] g_i \in B$,

since B is an ideal of FG. Also $[h, g_i^{-1}] \in H$ for $1 \leqslant i \leqslant n$, since $H \trianglelefteq G$.

Thus, $x - x^h \in B$ and

$$x - x^h = \sum_{i=2}^{n} \alpha_i (1 - [h, g_i^{-1}]) g_i$$ and by the choice of x, $x - x^h = 0$. Thus, $x = x^h$ for all $h \in H$.

Let $g \in \mathrm{Supp}(x)$. Then g has only a finite number of conjugates under conjugation by the elements of H. The number of conjugates of g is bounded by the length of x. Thus, the index in H of $\mathcal{C}_H(\mathrm{Supp}(x)) = \{h \in H : g^h = g$ for all g belonging to $\mathrm{Supp}(x)\}$ is finite. Hence, since H is divisible, $H = \mathcal{C}_H(\mathrm{Supp}(x))$.

Let $K = \langle H, \mathrm{Supp}(x) \rangle$. Then $H \leqslant Z_1(K)$, the centre of K and K/H is a subgroup of G/H. Thus, either $K = H$, or $K = G$. If $K = H$, $x \in B \cap FH$ and we get a contradiction to lemma 6.8. If $K = G$, G/H is cyclic group, $\langle Ht \rangle$ say, and $H \leqslant Z_1(G)$. Thus, $G = \langle H, t \rangle$ and clearly G is abelian. Thus FG is commutative and $B(FG) = J(FG) = \{0\}$ by the remarks following ~~lemma~~ Theorem 3.8.

Theorem 6.10. Let G be a group, H a normal $\mathfrak{A}$ subgroup of finite index in G with $G/H \in \mathfrak{A}$. Let F be an algebraically closed field of characteristic p where, if $p \neq 0$, G and G/H have no elements of order p. Then FG is Brown-McCoy semi-simple.

Proof. By lemma 6.8, $B \cap FH = \{0\}$. By theorem 3.8 $B = (B \cap FH)FG = \{0\}$.

We make the following conjecture.

Conjecture 1. If G is a soluble group with the minimal condition and F a field of characteristic p where, if $p \neq 0$, G and each factor in the derived series of G have no elements of order p. Then FG is Brown-McCoy semi-simple.

We feel that this conjecture is true since it it well known that a soluble group with the minimal condition is a finite soluble extension of a divisible abelian group.

The remainder of this section is taken up with the study of metabelian groups with a maximal abelian normal subgroup which is torsion-free.

Lemma 6.11. Let G be any metabelian group, H a maximal normal abelian subgroup of G such that H is torsion-free. If F is any field and I any two sided ~~characteristic~~ non zero ideal of FG, then $I \cap FH \neq \{0\}$.

Proof. Choose x to be any non zero element of I and T to be any transversal (with 1) to H in G. Then we can assume that

$$x = \sum_{i=1}^{n} \alpha_i g_i, \text{ where } n \geqslant 1,\ 0 \neq \alpha_i \in FH \text{ and}$$

the g_i's are distinct elements of T for $1 \leqslant i \leqslant n$, is the H-decomposition of x.

Choose $x \in I$ such that x is of minimal H-length among the non zero elements of I and we can assume that one of the g_i's, g_1 say, is 1.

Let $h \in H$. Then, since I is a two sided ideal of FG, $x - x^h \in I$.

However, $x - x^h = \sum_{i=2}^{n} \alpha_i (1 - [h, g_i^{-1}]) g_i$, since we assumed that $g_1 = 1$. However, $\alpha_i(1 - [h, g_i^{-1}])$

belongs to FH, for $i = 2, 3, \ldots, n$ since H is normal in G. Thus $x - x^h$ has smaller H-length than x and thus is zero. Thus, $\alpha_i(1 - [h, g_i^{-1}])$ is zero for $i = 2, 3, \ldots, n$ by uniqueness.

H is abelian torsion-free and hence can be ordered. Thus, FH has no zero divisors by theorem 4.1 (i). Since $\alpha_i \neq 0$, we must have $[h, g_i^{-1}] = 1$ for $i = 2, 3, \ldots, n$.

Let $C = \mathcal{C}_G(H)$. Then C is normal in G since H is normal in G. Also $\mathrm{Supp}(x) \leqslant C$. Choose $k \in C$. Then $x - x^k \in I$ and

$$x - x^k = \sum_{i=2}^{n} \alpha_i (1 - [k, g_i^{-1}]) g_i.$$

Since G is metabelian, $[k, g_i^{-1}] \in C \cap G'$.

However, $H(C \cap G')$ is an abelian normal subgroup of G containing H and hence must be equal to H.

Thus $x - x^k = 0$ and since FH has no zero divisors, $[k, g_i^{-1}] = 1$ for $i = 1, 2, \ldots, n$.

Hence $\mathrm{Supp}(x) \leqslant Z(C)$ where $Z(C)$ is the centre of C. However, $Z(C)$ contains H and is characteristic in C and hence normal in G. Thus $C = H$ and $x \in FH$. Thus $x \in I \cap FH$ as required.

Corollary 6.12. Let G be any metabelian group and F a field of characteristic p. Suppose that G has a maximal abelian normal subgroup H, of finite index in G such that H is torsion-free. Then if p non zero implies that p does not divide the order of G/H, then FG is Brown-McCoy semi-simple.

Proof. If $B(FG) \neq \{0\}$, then lemmas 6.10 and 6.8 give an immediate contradiction.

Lemma 6.11 however, can be used to construct metabelian groups whose group algebra over a field of characteristic p is Brown-McCoy semi-simple even when G has an element of order p.

We now define a class, $\mathfrak{X}$, of metabelian groups by G is an $\mathfrak{X}$ -group if and only if G has a maximal abelian normal subgroup H, such that H is torsion-free and of the form $Z \times \langle x \rangle$, where Z is the centre of G and $\langle x \rangle$ is normal in G.

The class $\mathfrak{X}$ is not empty since it contains $A = \langle a,b: a^b = a^{-1} \rangle$, $B = \langle a,b: a^b = a^{-1}, b^2 = 1 \rangle$.

In [27] Wallace has shown that FB is Jacobson semi-simple when F is the field of 2 elements. We shall show that if $G \in \mathfrak{X}$ then FG is Brown--McCoy semi-simple for any field F.

Lemma 6.13. Let G be any element of $\mathfrak{X}$, F any field and $I \neq 0$ any two sided ideal of FG. Then, $I \cap FH \cap Z^*$ is non zero where Z^* is the centre of FG.

Proof. By lemma 6.11, $I \cap FH \neq \{0\}$. Choose a to be any non zero element of the intersection. Then, a can be written uniquely in the form

$$a = \sum_{i=1}^{m} \beta_i x^{n(i)}, \quad m \geqslant 1, \quad \beta_i \in FZ \leqslant Z^*$$

and the n(i) distinct integers for i = 1, 2, ..., n.

Now there exists a $g \in G$ such that $x^g = x^{-1}$ since $\langle x \rangle$ is torsion-free, $x \notin Z$ and $\langle x \rangle$ is normal in G. Clearly $a + a^g \in I \cap FH$ since H is normal in G. However,

$$a + a^g = \sum_{i=1}^{m} \beta_i (x^{n(i)} + x^{-n(i)}) = b \text{ say.}$$

Clearly $b = b^h$ for all $h \in G$ and thus $b \in Z^*$ as required.

Theorem 6 .14. Let G be any $\mathfrak{X}$ -group and F any field. Then FG is Brown-McCoy semi-simple.

Proof. Assume not and let B denote the Brown--McCoy radical. Then by lemmas 6.2 and 6.13 $J(Z^*) \cap FH$ is non zero. Lemma 3.10 implies that $J(Z^* \cap FH)$ is non zero. However, H is abelian torsion-free and thus theorem 4.1 (iii) gives a contradiction.

4. On residually finite groups.

In this section we are concerned with residually finite groups and will determine sufficent conditions for the group algebra of a residually finite group to be Brown-McCoy semi-simple. This generalises the results of [27].

Let G be a residually finite group. Let $\{H_\lambda : \lambda \in \Lambda\}$ be a family of normal subgroups satisfying condition (a) and (b) of lemma 3.21. Then G/H_λ is finite for all $\lambda \in \Lambda$.

Theorem 6.15. Let G be a residually finite group and $\{H_\lambda : \lambda \in \Lambda\}$ the family of normal subgroups mentioned above. Let F be a field of characteristic p where, if $p \neq 0$, G/H_λ has no elements of order p for all $\lambda \in \Lambda$. Then FG is Brown-McCoy semi-simple.

Proof. Let B denote the Brown-McCoy radical of

FG. Then in the homomorphism from FG to $F(G/H_\lambda)$ $B + I_\lambda$ is mapped onto a B-regular ideal of $F(G/H_\lambda)$, which is contained in the Brown-McCoy radical of $F(G/H_\lambda)$. However, G/H_λ is finite and thus $B(F(G/H_\lambda)) = J(F(G/H_\lambda)) = \{0\}$ by the choice of F. Thus $B \leq I_\lambda$. This is true for all $\lambda \in \Lambda$, and thus $B \leq \bigcap_{\lambda \in \Lambda} I_\lambda = \{0\}$ by lemma 3.21. Thus, FG is Brown-McCoy semi-simple.

Corollary 6.16. Let G be any soluble group with the maximal condition and F any field of character--istic p, where if $p \neq 0$, G has no finite quotients whose order is divisible by p. Then FG is Brown-McCoy semi-simple.

Proof. This follows since $G \in R\mathfrak{F}$.

Corollary 6.17. Let F be any field and G any free group. Then FG is Brown-McCoy semi-simple.

Proof. By a result of Iwasawa, free groups are residually finite p-groups for any prime p. Choose p different to the characteristic of F.

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